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ON GOOD $\mathscr{L}_p$ SUBSPACES OF l_p

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ABSTRACT
THEOREM. Given $K > 1$ and $1 \leqq p < \infty$, there is $\lambda > 1$ so that every $\mathscr{L}_{p,\lambda}$ subspace of l_p is K-isomorphic to l_p.

0. Introduction and preliminaries

Let us recall that a separable Banach space X is called a $\mathscr{L}_p$ space if there is a sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ of finite dimensional subspaces with $\overline{\bigcup_{n=1}^{\infty} X_n} = X$ and a constant $\mu < \infty$ so that

$$d(X_n, l_p^{d(n)}) \leqq \mu, \qquad n = 1, 2, \cdots$$

where $d(n)$ denotes the dimension of X_n. More precisely we say that X is a $\mathscr{L}_{p,\lambda}$ space if the infimum of such μ's is no larger than λ. We refer to $\mathscr{L}_{p,\lambda}$ spaces with small value of λ as "good $\mathscr{L}_p$ spaces".

In his dissertation [10] M. Zippin proved that a space X with $\mu = 1$ is isometric to $L_p(\nu)$ for a suitable measure ν (see also [6]). J. Lindenstrauss and A. Pełczyński [7] proved that the same conclusion holds for X a $\mathscr{L}_{p,1}$ space. In view of precedents in functional analysis (see for example [2], [3], [4], [9], [11] and [12]), it is natural to ask whether this result extends by continuity to values of λ close to 1, i.e.

PROBLEM A. Is there a $\lambda_p > 1$ and a function ϕ_p from $(1, \lambda_p)$ to $(1, \infty)$ with $\lim_{\lambda \to 1+} \phi_p(\lambda) = 1$ so that every $\mathscr{L}_{p,\lambda}$ space is $\phi_p(\lambda)$-isomorphic to some $L_p(\nu)$ space?

A partial answer to this problem was given by M. Zippin [11] in the case $p = 1$. He gave a function $\phi_1(\lambda)$ for $1 < \lambda < \lambda_1$ so that every $\mathscr{L}_{p,\lambda}$ space which embeds isometrically in l_1 is $\phi_1(\lambda)$-isomorphic to l_1. However he had $\lim_{\lambda \to 1+} \phi_1(\lambda) = 2$.

In this paper we extend Zippin's result to all values of $1 \leqq p < \infty$ and give $\lim_{\lambda \to 1+} \phi_p(\lambda) = 1$.

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Zippin's construction [11] plays a central role in our proof. It is presented here in a streamlined and modified form.

We shall need two known results on sequences in $L_p(\nu)$ spaces.

PROPOSITION 1 ([4], [1], *see also* [5]). *Let* $1 \leqq p < \infty$, $p \neq 2$. *There is* $\lambda_p > 1$ *and function* $\delta_p(\lambda)$ *from* $(1, \lambda_p)$ *to* $(0,1)$ *with* $\lim_{\lambda \to 1^+} \delta_p(\lambda) = 0$ *so that for any* $1 < \lambda < \lambda_p$, *any (finite or infinite) sequence* $(x_1, x_2, \cdots)$ *in any* $L_p(\nu)$*-space, if the inequalities*

$$\lambda^{-1}\Big(\sum_i |\alpha_i|^p\Big)^{1/p} \leqq \Big\| \sum_i \alpha_i x_i \Big\| \leqq \lambda \Big(\sum_i |\alpha_i|^p\Big)^{1/p}$$

hold for all choices of scalars $(\alpha_1, \alpha_2, \cdots)$ *(with finitely many non-zeros), then there are disjoint* ν*-measurable sets* $A_1, A_2, \cdots$ *such that*

$$\| x_{i|_{\sim A_i}} \| < \delta_p(\lambda), \qquad \textit{for all } i.$$

PROOF. The proof of proposition 2.1 of [4], modified slightly to account for the fact that the x_i are no longer assumed to be unit vectors, gives

$$\Big\| \max_i |\alpha_i x_i| \Big\| \geqq \lambda^{-(p+2)/|p-2|} \Big(\sum_i |\alpha_i|^p\Big)^{1/p}$$

for all finitely non-zero choices of coefficients (α_i). Then the proof of proposition 2.2 of [4] provides us with functions ϕ_i in $L_\infty(\nu)$ with $\phi_i \geqq 0$, $\Sigma_i \phi_i \leqq 1$ a.e. so that

$$\int |x_i|^p \phi_i d\nu \geqq \lambda^{-p(p+2)/|p-2|}, \qquad \text{all } i.$$

Now if we take $A_i = [\phi_i > \frac{1}{2}]$, we will clearly have disjoint sets A_i, and a simple computation will give

$$\int_{\sim A_i} |x_i|^p \leqq 2\{\lambda^p - \lambda^{-p(p+2)/|p-2|}\} \equiv \delta_p(\lambda)^p$$

for all i.

PROPOSITION 2 (G. Schechtman [9]). *Given* $1 \leqq p < \infty$, $p \neq 2$ *there is* ε_p *and function* $a_p : (0, \varepsilon_p) \to (0,1)$ *so that* $a_p(\varepsilon) \to 0$ *as* $\varepsilon \to 0+$ *and for all* $0 < \varepsilon < \varepsilon_p$, *if* $(x_i) \subseteq L_p(\nu)$ *and* (A_i) *is a sequence of disjoint* ν*-measurable sets such that*

$$\text{(i)} \qquad (1-\varepsilon)\Big(\sum_i |\alpha_i|^p\Big)^{1/p} \leqq \Big\| \sum_i \alpha_i x_i \Big\| \leqq (1+\varepsilon)\Big(\sum_i |\alpha_i|^p\Big)^{1/p}$$

for all finitely non-zero (α_1)*'s, and*

$$\text{(ii)} \qquad \| x_{i|_{\sim A_i}} \| < \varepsilon \qquad \textit{for all } i,$$

then

$$\Big\| \sum_i \alpha_i x_{i|\sim A_i} \Big\| \leqq a_p(\varepsilon) \Big(\sum_i |\alpha_i|^p \Big)^{1/p}$$

for all finitely non-zero sequences of scalars (α_i).

We shall need the following perturbation lemma which is a simple consequence of Proposition 2:

LEMMA 3. *Let* $1 \leqq p < \infty$, $p \neq 2$, *assume that* ε *is small enough and let* (x_i), (y_i) *be two sequences in* $L_p(\nu)$ *such that*

$$\| x_i - y_i \| < \varepsilon \qquad \text{for each } i,$$

and such that for all finitely non-zero sequences of scalars (α_i),

$$(1-\varepsilon)\Big(\sum_i |\alpha_i|^p \Big)^{1/p} \leqq \Big\| \sum_i \alpha_i x_i \Big\| \leqq (1+\varepsilon)\Big(\sum_i |\alpha_i|^p \Big)^{1/p},$$

$$(1-\varepsilon)\Big(\sum_i |\alpha_i|^p \Big)^{1/p} \leqq \Big\| \sum_i \alpha_i y_i \Big\| \leqq (1+\varepsilon)\Big(\sum_i |\alpha_i|^p \Big)^{1/p}.$$

Then for all such sequences (α_i),

$$\Big\| \sum_i \alpha_i (x_i - y_i) \Big\| \leqq \eta \cdot \Big(\sum_i |\alpha_i|^p \Big)^{1/p},$$

where $\eta = \eta_p(\varepsilon) = 2a_p(\delta_p((1-\varepsilon)^{-1})) + 2\delta_p((1-\varepsilon)^{-1}) + 3\varepsilon$, *so in particular* $\eta_p(\varepsilon) \to 0$ *as* $\varepsilon \to 0+$.

PROOF. By Proposition 1 there are disjoint sets (A_i) and disjoint sets (B_i), all ν-measurable, so that

$$\| x_{i|\sim A_i} \| \leqq \delta, \qquad \text{all } i,$$

$$\| y_{i|\sim B_i} \| \leqq \delta, \qquad \text{all } i,$$

where $\delta = \delta_p\ ((1-\varepsilon)^{-1})$. From our assumption we get that for all i,

$$\| x_{i|\sim B_i} \| \leqq \| y_{i|\sim B_i} \| + \| x_i - y_i \| \leqq \delta + \varepsilon,$$

and similarly

$$\| y_{i|\sim A_i} \| \leqq \delta + \varepsilon.$$

Thus for all finitely non-zero sequences (α_i) of scalars,

$$\begin{aligned}\left\|\sum_i \alpha_i(x_i-y_i)\right\| &\leqq \left\|\sum_i \alpha_i x_{i|\sim A_i}\right\| + \left\|\sum_i \alpha_i y_{i|\sim B_i}\right\| + \left\|\sum \alpha_i x_{i|A_i} - \alpha_i y_{i|B_i}\right\| \\ &\leqq 2a_p(\delta)\left(\sum_i |\alpha_i|^p\right)^{1/p} + \left\|\sum_i \alpha_i x_{i|A_i\setminus B_i}\right\| \\ &\quad + \left\|\sum_i \alpha_i y_{i|B_i\setminus A_i}\right\| + \left\|\sum_i \alpha_i(x_i-y_i)_{|A_i\cap B_i}\right\| \qquad \text{by Proposition 2} \\ &\leqq 2a_p(\delta)\left(\sum_i |\alpha_i|^p\right)^{1/p} + \left(\sum_i \|\alpha_i x_{i|\sim B_i}\|^p\right)^{1/p} \\ &\quad + \left(\sum_i \|\alpha_i y_{i|\sim A_i}\|^p\right)^{1/p} + \left(\sum_i \|\alpha_i(x_i-y_i)\|^p\right)^{1/p} \qquad \text{by disjointness} \\ &\leqq \{2a_p(\delta)+2(\delta+\varepsilon)+\varepsilon\}\left(\sum_i |\alpha_i|^p\right)^{1/p}.\end{aligned}$$

In the case $p=1$ this is a special case of a trivial and well known perturbation lemma for equivalents of the usual l_1 basis. For $1<p<\infty$, $p\neq 2$, it can be shown that the requirements that *both* sequences be well equivalent to the usual l_p-basis and that the ambient space be an L_p-space cannot be dropped.

We use standard Banach space-theoretic notation as can be found in [8], for example. For a function f and set A, $f_{|A}$ is the function that equals f on A and equals 0 off A. For an infinite set M, $\mathbf{P}_\infty(M)$ will denote the family of all the infinite subsets of M.

1. Proof of the theorem stated in the abstract

We fix $1\leqq p<\infty$, $p\neq 2$. To simplify notation we choose a version of the moduli mentioned in the introduction so that

(1) $$\delta(\lambda)=\delta_p(\lambda)\geqq \lambda-1, \qquad \text{all } 1\leqq\lambda<\lambda_p,$$

(2) $$a(\varepsilon)=a_p(\varepsilon)\geqq\varepsilon, \qquad \text{all } 0<\varepsilon<\varepsilon_p,$$

(3) $$\eta(\varepsilon)=\eta_p(\varepsilon)\geqq\varepsilon, \qquad \text{all } 0<\varepsilon<\varepsilon_p.$$

We choose $\varepsilon>0$ such that

(4) $$\{1-2\eta(5\eta(2\varepsilon))\}^{-1}<K,$$

as well as

(5) $$(1+\varepsilon)^p<3/2 \quad\text{and}\quad \varepsilon<1/10.$$

Next we choose $\delta > 0$ such that

$$\delta \leqq a(\delta) < \varepsilon^2/10 \tag{6}$$

and finally find $\mu > 1$ such that

$$\delta(\mu) \leqq \delta \qquad (\text{so } \mu < 1+\delta). \tag{7}$$

We shall show that any $\lambda < \mu$ will satisfy the statement of the theorem.

Fix a $\mathscr{L}_{p,\lambda}$ subspace X of l_p. Since $\lambda < \mu$, there are finite-dimensional subspaces

$$X_1 \subset X_2 \subset X_3 \subset \cdots \tag{8}$$

such that

$$\overline{\bigcup_{n=1}^{\infty} X_n} = X \quad \text{and} \quad d(X_n, l_p^{d(n)}) \leqq \mu, \qquad \text{for all } n. \tag{9}$$

For each X_n we can find a basis $(x_1^n, x_2^n, \cdots, x_{d(n)}^n)$ such that

$$\mu^{-1}\left(\sum_{j=1}^{d(n)} |\alpha_j|^p\right)^{1/p} \leqq \left\| \sum_{j=1}^{d(n)} \alpha_j x_j^n \right\| \leqq \mu \left(\sum_{j=1}^{d(n)} |\alpha_j|^p\right)^{1/p} \tag{10}$$

for all choices of scalars $(\alpha_j, j \leqq d(n))$.

For each n we choose by Proposition 1 disjoint subsets $A_1^n, A_2^n, \cdots, A_{d(n)}^n$ of $\mathbf{N}$ so that

$$\| x^n_{j|\sim A_j^n} \| < \delta(\mu) \leqq \delta \qquad \text{for all } j \leqq d(n). \tag{11}$$

We may and shall assume that all the sets A_j^n are *finite*. It might be helpful to keep in mind a picture of the double array $(x_j^n, n \in \mathbf{N}, j \leqq d(n))$ with the row $(x_1^n, x_2^n, \cdots, x_{d(n)}^n)$ as its n'th level.

With each vector x_j^n on the n'th level we associate a set of indices on the k'th level $(k > n)$ as follows:

$$\begin{aligned} &C_j^{n,k} = \{h \leqq d(k);\ \| x^k_{h|A_j^n \cap A_h^k} \| > (1-\varepsilon^p)^{1/p}\}, \\ &\text{for } 1 \leqq j \leqq d(n), \qquad n < k. \end{aligned} \tag{12}$$

Also for all $n < k$ and $j \leqq d(n)$ we set

$$z_h^{n,k} = x^k_{h|A_j^n \cap A_h^k} \qquad \text{for all } h \in C_j^{n,k}. \tag{13}$$

We now claim that

1°. *For any $n < k$, the sets $C_1^{n,k}, \ldots, C_{d(n)}^{n,k}$ are disjoint.* (This shows that the definition in (13) is a proper one.)

2°. *For any $n<k$ and $j\leqq d(n)$,*

$$d(x_j^n,[x_h^k, h\in C_j^{n,k}])<\varepsilon.$$

3°. *For any $n<k$ and $j\leqq d(n)$, the supports of the functions $(z_h^{n,k}, h\in C_j^{n,k})$ are disjoint and contained in A_j^n.*

4°. *For all $n<k$ and $h\in\bigcup_{j=1}^{d(n)}C_j^{n,k}$,*

$$\|x_h^k - z_h^{n,k}\|\leqq\varepsilon \qquad \text{and} \tag{14}$$

$$\mu\geqq\|z_h^{n,k}\|\geqq(1-\varepsilon^p)^{1/p}. \tag{15}$$

PROOF OF THE CLAIM. 1° holds since any $h\in C_i^{n,k}\cap C_j^{n,k}$ for $1\leqq i\neq j\leqq d(n)$ would have to satisfy by (10) and (12) the inequality

$$\mu^p\geqq\|x_h^k\|^p\geqq\|x_{h|A_i^n}^k\|^p+\|x_{h|A_j^n}^k\|^p\geqq 2(1-\varepsilon^p)$$

which is absurd by (5), (6) and (7).

3° and 4° are obvious from the definitions (12) and (13).

To prove 2°, choose $n<k$ and $j\leqq d(n)$. Since $X_n\subset X_k$, we can find scalars $\beta_1,\beta_2,\cdots,\beta_{d(k)}$ so that

$$x_j^n=\sum_{h=1}^{d(k)}\beta_h x_h^k. \tag{16}$$

We introduce the auxiliary functions

$$y_h^k = x_{h|A_h^k}^k,\qquad 1\leqq h\leqq d(k).$$

For any $h\notin C_j^{n,k}$ we have by (12)

$$\begin{aligned}\varepsilon^p &\leqq\|x_{h|\sim(A_j^n\cap A_h^k)}^k\|^p\\ &=\|x_{h|(\sim A_j^n)\cap A_h^k}^k\|^p+\|x_{h|\sim A_h^k}^k\|\\ &\leqq\|y_{h|\sim A_j^n}^k\|^p+\delta^p \qquad\text{by (11).}\end{aligned}$$

This gives

$$\begin{aligned}(\varepsilon^p-\delta^p)^{1/p}\Big(\sum_{h\notin C_j^{n,k}}|\beta_h|^p\Big)^{1/p} &\leqq\Big(\sum_{h=1}^{d(k)}\|\beta_h y_{h|\sim A_j^n}^k\|^p\Big)^{1/p}\\ &=\Big\|\sum_{h=1}^{d(k)}\beta_h y_{h|\sim A_j^n}^k\Big\| \qquad\text{by disjointness}\\ &\leqq\|x_{j|\sim A_j^n}^n\|+\Big\|\sum_{h=1}^{d(k)}\beta_h(x_h^k-y_h^k)_{|\sim A_j^n}\Big\| \qquad\text{by (16)}\end{aligned}$$

$$\leqq \delta + \left(\sum_{h=1}^{d(k)} |\beta_h|^p \right)^{1/p} a(\delta) \qquad \text{by (11) and Proposition 2}$$

$$\leqq \delta + \mu a(\delta) \| x_j^n \| \leqq \delta + \mu^2 a(\delta) \qquad \text{by (10).}$$

This, together with (10) and (16), gives

$$d(x_j^n, [x_h^k, h \in C_j^{n,k}]) \leqq \left\| \sum_{h \notin C_j^{n,k}} \beta_h x_h^k \right\|$$

$$\leqq \mu \left(\sum_{h \notin C_j^{n,k}} |\beta_h|^p \right)^{1/p} \leqq \frac{\mu^2 a(\delta) + \delta}{(\varepsilon^p - \delta^p)^{1/p}}$$

$$\leqq \frac{5a(\delta)}{\varepsilon/2} = 10a(\delta)/\varepsilon < \varepsilon \qquad \text{by (2), (6) and (7),}$$

which concludes the proof of the claim.

Our next step is to pass to a subsequence $n(1) < n(2) < \cdots$ of the set of levels, on which the sets $C_j^{n,k}$ and the functions $z_h^{n,k}$ will behave in a mutually compatible way, namely, if we set

$$D_1 = \{1, 2, \cdots, d(n(1))\} \tag{17}$$

and, for each $r > 1$, set

$$\begin{cases} D'_r = \bigcup \{C_j^{n(s),n(r)}, s < r, j \in D_s\}, & \text{and} \\ D_r = \{1, 2, \cdots, d(n(r))\} \setminus D'_r, \end{cases} \tag{18}$$

then the following properties will hold:

5°. *For any $r \geqq 1$ and any $j \in D_r$, the sets $C_j^{n(r),n(s)}$, $s = r+1, r+2, \cdots$ have all the same number of elements.*

6°. *For any $r \geqq 1$, any $j \in D_r$, and any s, $s' > r$, to any $h \in C_j^{n(r),n(s)}$ there corresponds $h' \in C_j^{n(r),n(s')}$ such that*

$$\| z_h^{n(r),n(s)} - z_{h'}^{n(r),n(s')} \| < \varepsilon.$$

To get this, we define by induction a sequence $n(1) < n(2) < \cdots$ of positive integers, a sequence $K_1 \supset K_2 \supset \cdots$ in $\mathbf{P}_\infty(\mathbf{N})$, and the sets $D_1, D_2, \cdots$ as specified by (17) and (18) so that

(i) $n(r) = (\min K_r) \notin K_{r+1}$, *all* $r \in \mathbf{N}$,

(ii) *for each $r \in \mathbf{N}$ and each $j \in D_r$, the sets $C_j^{n(r),k}$, $k \in K_{r+1}$ have all the same number of elements,*

and

(iii) *for all* $r \in \mathbf{N}$, $j \in D_r$ *and* k, $k' \in K_{r+1}$, *for each* $h \in C_j^{n(r),k}$ *there is* $h' \in C_j^{n(r),k'}$ *so that*

$$\| z_h^{n(r),k} - z_{h'}^{n(r),k'} \| < \varepsilon.$$

The conditions (i)–(iii) clearly imply that 5° and 6° are satisfied.

We start the inductive construction with $K_1 = \mathbf{N}$, $n(1) = 1$, and $D_1 = \{1, 2, \cdots, d(n(1))\}$. For the inductive step we take $s > 1$ and assume that

$$n(1) < n(2) < \cdots < n(s) \quad \text{and} \quad K_1 \supset K_2 \supset \cdots \supset K_s$$

have been defined so that the conditions (i)–(iii) hold for all $1 \leqq r \leqq s - 1$ (under the conventions (17) and (18)), and moreover, $n(s) = \min K_s$. Let $K'_s = K_s \setminus \{n(s)\}$.

Fix for a while $j \in D_s$. For each $k > n(s)$ the functions $(z_h^{n(s),k},\ h \in C_j^{n(s),k})$ are non-zero, with disjoint supports all contained in the fixed finite set $A_j^{n(s)}$ (see 3°, 4°), Consequently, we have

$$| C_j^{n(s),k} | \leqq | A_j^{n(s)} | \qquad \text{for all } k > n(s).$$

Since we have such a bound, independent of k, for each $j \in D_s$, we can find $K''_s \in \mathbf{P}_\infty(K'_s)$ and integers $c_j \leqq A_j^{n(s)}$, $j \in D_s$ so that

$$| C_j^{n(s),k} | = c_j, \qquad \text{all } k \in K''_s, \quad j \in D_s.$$

Now for each $j \in D_s$ and $k \in K''_s$, the c_j-tuple $(z_h^{n(s),k},\ h \in C_j^{n(s),k})$ belongs to the compact set $\{\mu \text{ Ball } (l_p(A_j^{n(s)}))\}^{c_j}$. (For definiteness we consider the indices $h \in C_j^{n(s),k}$ in their natural order.) Therefore we can find $K_{s+1} \in \mathbf{P}_\infty(K_s)$ so that for each $j \in D_s$, for all k, $k' \in K_{s+1}$, and each $h \in C_j^{n(s),k}$ we have

$$\| z_h^{n(s),k} - z_{h'}^{n(s),k'} \| < \varepsilon$$

where h' is the index corresponding to h under the natural orderings of $C_j^{n(s),k}$ and of $C_j^{n(s),k'}$.

Finally we set $n(s+1) = \min K_{s+1}$ and now the induction hypothesis holds with s replaced by $s + 1$, so the inductive construction is complete.

As a consequence of 6° we obtain the following property:

7°. *For any* $q < r < s$ *and for any* $i \in D_q$ *and* $j \in D_r$ *we have*

$$C_i^{n(q),n(s)} \cap C_j^{n(r),n(s)} = \varnothing.$$

To see this, assume that $h \in C_i^{n(q),n(s)} \cap C_j^{n(r),n(s)}$. Then by 6° there is $h' \in C_i^{n(q),n(r)}$ so that

$$\| z_h^{n(q),n(s)} - z_{h'}^{n(q),n(r)} \| \leqq \varepsilon. \tag{19}$$

Now we have two indices h' and j on the $n(r)$'th level. Since $h' \in D'_r$ and $j \in D_r$, we have $h' \neq j$, and so

$$z_{h'}^{n(q),n(r)}{}_{|A_j^{n(r)}} = 0. \tag{20}$$

Consequently,

$$\begin{aligned}\| x_h^{n(s)}{}_{|A_j^{n(r)}} \| &\leqq \| x_h^{n(s)} - z_{h'}^{n(q),n(r)} \| \qquad \text{by (20)}\\ &\leqq \| x_h^{n(s)} - z_h^{n(q),n(s)} \| + \| z_h^{n(q),n(s)} - z_{h'}^{n(q),n(r)} \| \\ &\leqq \varepsilon + \varepsilon = 2\varepsilon \qquad \text{by (14) and (19).}\end{aligned}$$

But $h \in C_j^{n(r),n(s)}$, so by definition (12),

$$2\varepsilon \geqq \| x_h^{n(s)}{}_{|A_j^{n(r)}} \| \geqq (1-\varepsilon^p)^{1/p},$$

which contradicts (5) and thus proves the validity of 7°.

In view of 5°, 6° and 7° we may (and shall) assume, without loss of generality, that

5°°. *For all* $r \in \mathbf{N}$, $j \in D_r$ *and* $s, s' > r$,

$$C_j^{n(r),n(s)} = C_j^{n(r),n(s')} \equiv C_j^{n(r)},$$

6°°. *For all* r *all* $s, s' > r$ *and all* $h \in \bigcup_{j \in D_r} C_j^{n(r)}$, *we have*

$$\| z_h^{n(r),n(s)} - z_h^{n(r),n(s')} \| < \varepsilon.$$

This can be done by permuting the elements in each consecutive row, and adjusting all the definitions (of $C_j^{n,k}$, etc.) according to the new ordering.

Finally, using again the precompactness of the set $(z_h^{n(r),n(s)}, s > r)$ for each r, and each $h \in \bigcup_{j \in D_r} C_j^{n(r)}$, together with a diagonal process, we can find a subsequence $s_1 < s_2 < \cdots$ so that for any r and any $h \in \bigcup_{j \in D_r} C_j^{n(r)}$, the limit

$$u_h^r = \lim_{t \to \infty} z_h^{n(r),n(s_t)} \tag{21}$$

exists in norm. This definition and 6°° give us

$$\| z_h^{n(r),n(s)} - u_h^r \| \leqq \varepsilon \quad \text{for all } r, \text{ all } s > r \text{ and } h \in \bigcup_{j \in D_r} C_j^{n(r)} \tag{22}$$

We introduce the index set

$$\Delta = \left\{ (r,h);\ r \in \mathbf{N}, h \in \bigcup_{j \in D_r} C_j^{n(r)} \right\},$$

and consider the family $\{u'_h, (r,h) \in \Delta\}$. For any given $s > 1$, the functions $(z_h^{n(r),n(s)}, (r,h) \in \Delta, r < s)$ are disjointly supported (on the sets $A_h^{n(s)}$), so passing to limit on $s_t \to \infty$ we obtain that the family $(u'_h, (r,h) \in \Delta)$ is disjointly supported (in view of the fact that disjointness on support is an isometric invariant in $L_p(\nu)$, $p \neq 2$). Thus the space $U = [u'_h, (r,h) \in \Delta]$ is isometric to l_p. Also, by (15),

$$1 - \varepsilon \leqq (1 - \varepsilon^p)^{1/p} \leqq \| u'_h \| \leqq \mu \leqq 1 + \delta < 1 + \varepsilon,$$

and so we have

$$(23) \qquad (1-\varepsilon)\Big(\sum_{(r,h)\in\Delta} |\alpha'_h|^p\Big)^{1/p} \leqq \Big\| \sum_{(r,h)\in\Delta} \alpha'_h u'_h \Big\| \leqq (1+\varepsilon)\Big(\sum_{(r,h)\in\Delta} |\alpha'_h|^p\Big)^{1/p}$$

for all finitely non-zero sets of coefficients (α'_h).

We now claim that

8°. *For all $x \in X$, we have*

$$d(x, U) \leqq 2\eta(5\eta(2\varepsilon)) \| x \|,$$

and

9°. *For all $u \in U$, we have*

$$d(u, X) \leqq 2\eta(2\varepsilon) \| u \|.$$

Once 8° and 9° are proved, we can estimate $d(X, U)$ as follows: Let P be a norm 1 projection of l_p onto U. Choose any

$$\alpha > 2\eta(5\eta(2\varepsilon)) \quad \text{and} \quad 1 > \beta > 2\eta(2\varepsilon).$$

(Note that by (4), $2\eta(2\varepsilon) < 1$.)

For each $x \in X$ there is $u \in U$ with $\| u - x \| \leqq \alpha \| x \|$, and so

$$\| (I - P)x \| = \| (I - P)(x - u) \| \leqq 2\alpha \| x \|.$$

Thus $\| (I - P)_{|X} \| \leqq 2\alpha$ and hence by Neuman's series, $P_{|X}$ is invertible and $\| (P_{|X})^{-1} \| \leqq 1/(1 - 2\alpha)$.

On the other hand $PX \subseteq U$ and for any $u \in U$ there is $x \in X$ with $\| u - x \| \leqq \beta \| u \|$. Thus $d(u, PX) \leqq \| u - Px \| = \| P(u - x) \| \leqq \beta \| u \|$. Since $\beta < 1$ and u is an arbitrary element of U, a well known consequence of the Hahn–Banach separation theorem implies that PX is actually the whole space U.

Thus,

$$d(X, l_p) = d(X, U) \leqq \| P_{|X} \| \; \| (P_{|X})^{-1} \| \leqq \frac{1}{1 - 2\alpha}.$$

Taking the infimum over all $\alpha > 2\eta(5\eta(2\varepsilon))$ we obtain therefore that

$$d(X, l_p) \leqq \{1 - 2\eta(5\eta(2\varepsilon))\}^{-1} < K,$$

which completes the proof of the theorem.

To prove the claim 8° it is enough to consider x in $\bigcup_r X_{n(r)}$, so we take $r \geqq 1$ and choose any

$$x = \sum_{j=1}^{d(n(r))} \alpha_j x_j^{n(r)}.$$

We shall define a block basis $(v_j, j \leqq d(n(r)))$ of $(u_h^r, (r,h) \in \Delta)$ and use it to estimate $d(x, U)$.

For each $j \in D_r'$ there is a (unique) $q < r$ and $i \in D_q$ with $j \in C_i^{n(q)} = C_i^{n(q),n(r)}$. We set $v_j = u_j^q$, and obtain, by (14) and (22),

$$\text{(24)} \quad \begin{cases} \| x_j^{n(r)} - v_j \| \leqq \| x_j^{n(r)} - z_j^{n(q),n(r)} \| + \| z_j^{n(q),n(r)} - u_j^q \| \\ \leqq \varepsilon + \varepsilon = 2\varepsilon \qquad \text{for all } j \in D_r'. \end{cases}$$

For each $j \in D_r$ we find by 2° coefficients $(\beta_h^j, h \in C_j^{n(r)})$ so that

$$\text{(25)} \quad \Big\| x_j^{n(r)} - \sum_{h \in C_j^{n(r)}} \beta_h^j x_h^{n(r+1)} \Big\| \leqq \varepsilon.$$

For each index h in the above sum, we have

$$\text{(26)} \quad \begin{cases} \| x_h^{n(r+1)} - u_h^r \| \leqq \| x_h^{n(r+1)} - z_h^{n(r),n(r+1)} \| + \| z_h^{n(r),n(r+1)} - u_h^r \| \\ \leqq 2\varepsilon \qquad \text{by (14) and (22).} \end{cases}$$

By (10) and (23) both $(x_h^{n(r+1)}, h \in C_j^{n(r)})$ and $(u_h^r, h \in C_j^{n(r)})$ are close enough to the usual l_p-basis of the proper dimension so by Lemma 3 we get from (26) that

$$\Big\| \sum_{h \in C_j^{n(r)}} \beta_h^j x_h^{n(r+1)} - \sum_{h \in C_j^{n(r)}} \beta_h^j u_h^r \Big\| \leqq \eta(2\varepsilon) \Big(\sum_{h \in C_j^{n(r)}} |\beta_h^j|^p \Big)^{1/p}$$

$$\leqq \mu\eta(2\varepsilon) \Big\| \sum_{h \in C_j^{n(r)}} \beta_h^j x_h^{n(r+1)} \Big\| \leqq 4\eta(2\varepsilon).$$

This together with (25) gives

$$\text{(27)} \quad \| x_j^{n(r)} - v_j \| \leqq \varepsilon + 4\eta(2\varepsilon) \leqq 5\eta(2\varepsilon)$$

where we set

$$v_j = \sum_{h \in C_j^{n(r)}} \beta_h^j u_h^r, \qquad j \in D_r.$$

For each such j,

$$\begin{aligned} \|v_j\| &\leqq (1+\varepsilon)\Big(\sum_{h\in C_j^{n(r)}} |\beta_h^j|^p\Big)^{1/p} && \text{by (23)}\\ &\leqq (1+\varepsilon)\mu\Big\|\sum_{h\in C_j^{n(r)}} \beta_h^j x_h^{n(r+1)}\Big\| && \text{by (10)}\\ &\leqq (1+\varepsilon)\mu(\mu+\varepsilon) && \text{by (25)}\\ &\leqq (1+\varepsilon)^2(1+2\varepsilon)\leqq(1+5\varepsilon) && \text{by (7) and (5).}\end{aligned}$$

Similarly $\|v_j\|\geqq(1-5\varepsilon)$, and so

$$\begin{aligned}(1-5\varepsilon)\Big(\sum_{j=1}^{d(n(r))}|\gamma_j|^p\Big)^{1/p} &\leqq \Big\|\sum_{j=1}^{d(n(r))}\gamma_j v_j\Big\|\\ &\leqq(1+5\varepsilon)\Big(\sum_{j=1}^{d(n(r))}|\gamma_j|^p\Big)^{1/p} \quad \text{for all } (\gamma_j).\end{aligned} \tag{28}$$

Since $\eta(2\varepsilon)\geqq 2\varepsilon>\mu-1$, we can apply Lemma 3 to the sequences (v_j), $(x_j^{n(r)})$ with ε replaced by $5\eta(2\varepsilon)$, and get by (10), (24), (27), (28) and by the definition of x that

$$\begin{aligned} d(x,U) &\leqq \Big\|\sum_{j=1}^{d(n(r))}\alpha_j x_j^{n(r)}-\sum_{j=1}^{d(n(r))}\alpha_j v_j\Big\|\\ &\leqq \eta(5\eta(2\varepsilon))\Big(\sum_{j=1}^{d(n(r))}|\alpha_j|^p\Big)^{1/p}\\ &\leqq \mu\eta(5\eta(2\varepsilon))\|x\|,\end{aligned}$$

which concludes the proof of claim 8°.

The proof of claim 9° is similar but somewhat simpler. It is of course enough to consider only elements of the form

$$u=\sum_{\substack{(q,h)\in\Delta\\ q\leqq r}}\alpha_h^q u_h^q. \tag{29}$$

Choose $s>r$, and note that for each $q\leqq r$ and $h\in\bigcup_{j\in D_q}C_j^{n(q),n(s)}$,

$$\begin{aligned}\|x_h^{n(s)}-u_h^q\| &\leqq \|x_h^{n(s)}-z_h^{n(q),n(s)}\|+\|z_h^{n(q),n(s)}-u_h^q\|\\ &\leqq 2\varepsilon \qquad \text{by (14) and (22).}\end{aligned}$$

This, together with (10), (23) and (29), gives by Lemma 3 that

$$\left\| u - \sum_{q=1}^{r} \sum_{j \in D_q} \sum_{h \in C_j^{n(q),n(s)}} \alpha_h^q x_h^{n(s)} \right\|$$

$$\leqq \eta(2\varepsilon)\Bigg(\sum_{\substack{(q,h)\in\Delta \\ q \leqq r}} |\alpha_h^q|^p \Bigg)^{1/p}$$

$$\leqq (1-\varepsilon)^{-1}\eta(2\varepsilon)\| u \| \leqq 2\eta(2\varepsilon)\| u \|, \qquad \text{by (23) and (5),}$$

which proves claim 9°, and completes the proof of the theorem.

REMARK. In the case $p = 1$ one can simplify the proof and obtain an explicit basis for X, namely take as basis the sequence

$$\Bigg(x_h^{n(r+1)};\ h \in \bigcup_{j \in D_r} C_j^{n(r),n(r+1)}, r \geqq 1 \Bigg).$$

The fact that this sequence is equivalent to the usual l_1-basis can be seen either by considering it as a perturbation of the corresponding family (u_h^r) or directly by shifting each finite combination to a far enough level in the array $(x_h^{n(r)})$.

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